

# Quantum phase transitions in bilayer quantum Hall systems

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We construct a quantum Ginsburg-Landau theory to study the quantum phase transition from the excitonic superfluid (ESF) to a possible pseudo-spin density wave (PSDW) at some intermediate distances driven by the magneto-roton minimum collapsing at a finite wavevector. We analyze the properties of the PSDW and explicitly show that a square lattice is the favorite lattice. We suggest that correlated hopping of vacancies in the active and passive layers in the PSDW state leads to very large and temperature dependent drag consistent with the experimental data. Comparisons with previous microscopic numerical calculations are made. Further experimental implications are given.

**Introduction.** Spin-polarized Bilayer Quantum Hall systems at total filling factor  $\nu_T = 1$  have been under enormous experimental and theoretical investigations over the last decade [1, 2]. When the interlayer separation  $d$  is sufficiently large, the bilayer system decouples into two separate compressible  $\nu = 1/2$  Fermi Liquid (FL) layers. However, when  $d$  is smaller than a critical distance, even in the absence of interlayer tunneling, the system undergoes a quantum phase transition into a novel spontaneous interlayer coherent incompressible phase which is an excitonic superfluid state (ESF) in the pseudospin channel [1, 2, 3, 4]. Although there are very little dissipations in both the ESF and FL, the experiment [5] discovered strong enhancement of drag and dissipations in an intermediate distance regime. One of the outstanding problems in BLQH is to understand novel phases and quantum phase transitions as the distance between the two layers is changed. If the experimental observations indicate that there is an intermediate phase between the two phases remains controversial. Even it exists, different scenarios are proposed for the nature of the intermediate phase [6, 7, 8, 9]. Using Hartree-Fock (HF) in the Lowest Landau Level (LLL) or trial wavefunctions approximations, many authors [6] proposed different kinds of translational symmetry breaking ground states as candidates of the intermediate state. In this paper, we develop a quantum Ginsburg-Landau (QGL) theory [10] to study the nature of the intermediate phase. We propose there are two critical distances  $d_{c1} < d_{c2}$  and three phases as the distance increases. When  $0 < d < d_{c1}$ , the system is in the phase ordered ESF state which breaks the internal  $U(1)$  symmetry, when  $d_{c1} < d < d_{c2}$ , the system is in a pseudo-spin density wave (PSDW) state which breaks the translational symmetry, there is a first order transition at  $d_{c1}$  driven by magneto-roton minimum collapsing at a finite wavevector in the pseudo-spin channel. When  $d_{c2} < d < \infty$ , the system becomes two weakly coupled  $\nu = 1/2$  Composite Fermion Fermi Liquid (FL) state. There is also a first order transition at  $d = d_{c2}$ . However, disorders could smear the two first order transitions into two second order transitions. We construct a quantum Ginzburg-Landau

theory to describe the ESF to the PSDW which break the two completely different symmetries and analyze in detail the properties of the PSDW. By using the QGL, we explicitly show that a square lattice is the favorite lattice. The correlated hopping of vacancies in the active and passive layers in the PSDW state leads to very large and temperature dependent drag consistent with the experimental data in [5]. Recently, the effects of small imbalance above the PSDW were studied in [11] and found to explain the experimental observations nicely in [12].

**Formalism in phase representation.** Consider a bi-layer system with  $N_1$  ( $N_2$ ) electrons in top (bottom) layer and with interlayer distance  $d$  in the presence of magnetic field  $\vec{B} = \nabla \times \vec{A}$ :

$$\begin{aligned} H &= H_0 + H_{int} \\ H_0 &= \int d^2x c_\alpha^\dagger(\vec{x}) \frac{(-i\hbar\vec{\nabla} + \frac{e}{c}\vec{A}(\vec{x}))^2}{2m} c_\alpha(\vec{x}) \\ H_{int} &= \frac{1}{2} \int d^2x d^2x' \delta\rho_\alpha(\vec{x}) V_{\alpha\beta}(\vec{x} - \vec{x}') \delta\rho_\beta(\vec{x}') \quad (1) \end{aligned}$$

where electrons have *bare* mass  $m$  and carry charge  $-e$ ,  $c_\alpha, \alpha = 1, 2$  are electron operators in top and bottom layers,  $\delta\rho_\alpha(\vec{x}) = c_\alpha^\dagger(\vec{x})c_\alpha(\vec{x}) - \bar{\rho}_\alpha, \alpha = 1, 2$  are normal ordered electron densities on each layer. The intralayer interactions are  $V_{11} = V_{22} = e^2/\epsilon r$ , while interlayer interaction is  $V_{12} = V_{21} = e^2/\epsilon\sqrt{r^2 + d^2}$  where  $\epsilon$  is the dielectric constant. For simplicity, we only discuss the balanced case in this paper. The effects of imbalance were discussed in detail in [10, 11].

Performing a singular gauge transformation  $\phi_a(\vec{x}) = e^{i\int d^2x' \arg(\vec{x}-\vec{x}')\rho(\vec{x}')} c_a(\vec{x})$  where  $\rho(\vec{x}) = c_1^\dagger(\vec{x})c_1(\vec{x}) + c_2^\dagger(\vec{x})c_2(\vec{x})$  is the total density of the bi-layer system. We can transform the Hamiltonian Eqn.1 into a Lagrangian of the Composite Boson  $\phi_a$  coupled to a Chern-Simon gauge field  $a_\mu$  [10]. We can write the two bosons in terms of magnitude and phase  $\phi_a = \sqrt{\bar{\rho}_a + \delta\rho_a} e^{i\theta_a}$ , then after absorbing the external gauge potential  $\vec{A}$  into  $\vec{a}$ , we get

the Lagrangian in the Coulomb gauge [10]:

$$\begin{aligned} \mathcal{L} = & i\delta\rho_+(\frac{1}{2}\partial_\tau\theta_+ - a_0) + \frac{\bar{\rho}}{2m}[\frac{1}{2}\nabla\theta_+ - \vec{a}]^2 \\ & + \frac{1}{2}\delta\rho_+V_+(\vec{q})\delta\rho_+ - \frac{i}{2\pi}a_0(\nabla \times \vec{a}) \\ & + \frac{i}{2}\delta\rho_-\partial_\tau\theta_- + \frac{\bar{\rho}}{2m}(\frac{1}{2}\nabla\theta_-)^2 + \frac{1}{2}\delta\rho_-V_-(\vec{q})\delta\rho_- \end{aligned} \quad (2)$$

where  $\delta\rho_\pm = \delta\rho_1 \pm \delta\rho_2$ ,  $\theta_\pm = \theta_1 \pm \theta_2$ , they satisfy commutation relations  $[\delta\rho_\alpha(\vec{x}), \theta_\beta(\vec{x}')] = 2i\hbar\delta_{\alpha\beta}\delta(\vec{x} - \vec{x}')$ ,  $\alpha, \beta = \pm$ .  $\bar{\rho} = \bar{\rho}_1 + \bar{\rho}_2$ ,  $V_\pm = \frac{V_{11} \pm V_{12}}{2}$ .

It was shown in [10] that the functional form in the spin sector in Eqn.2 achieved from the CB theory is the same as the microscopic LLL+HF approximation achieved in [2]. However, the spin stiffness  $\frac{\bar{\rho}}{2m}$  and the  $V_-(q)$  in Eqn.2 should be replaced by the effective ones calculated by the LLL+HF approximation:  $\rho_E$  and  $V_E(q) = a - bq + cq^2$  where the non-analytic term is due to the long-range Coulomb interaction,  $a \sim d^2$ ,  $b \sim d^2$ , but  $c$  remains a constant at small distances. In the ESF state [10], it is convenient to integrate out  $\delta\rho_-$  in favor of the phase field  $\theta_-$ .

$$\mathcal{L}_s = \frac{1}{2V_E(\vec{q})}(\frac{1}{2}\partial_\tau\theta_-)^2 + \rho_E(\frac{1}{2}\nabla\theta_-)^2 \quad (3)$$

where the dispersion relation of the NGM including higher orders of momentum can be extracted:

$$\omega^2 = [2\rho_EV_E(\vec{q})]q^2 = q^2(a - bq + cq^2) \quad (4)$$

**Instability driven by the collapsing of magneto-roton minimum.** As shown in [16] in the context of possible supersolid in Helium 4, the advantage to extend the dispersion relation beyond the leading order is that the QGL action can even capture possible phase transitions between competing orders due to competing interactions on microscopic length scales. In the following, we study the instability of the ESF state as the distance increases. It can be shown that the dispersion curve Eqn.4 and  $V_E(q)$  indeed has the shape shown in Fig.1a [13] and Fig.1b respectively.

The phenomenon of the collapsing of the magneto-roton minimum as the distance increases was clearly demonstrated in the numerical calculations in the LLL+HF approaches [3, 14] and detected by inelastic light scattering [15]. It was estimated that  $q_0l \sim 1$ , so the lattice constant of the PSDW is of the same order of magnetic length  $l$  which is  $\sim 100\text{\AA}$ . The critical distance  $d_{c1}$  is also of the same order of the magnetic length. In reality, the instability happens before the minimum touches zero.

**Effective action in the dual density representation and a Feymann relation in the pseudo-spin channel.** Because the original instability comes from the density-density interaction, it is convenient to integrate

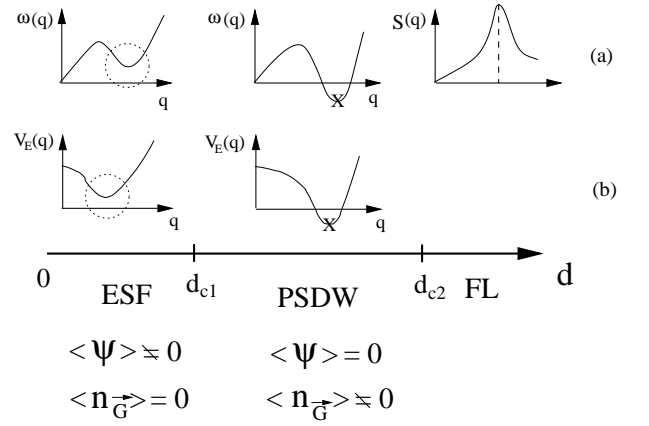


FIG. 1: The zero temperature phase diagram in the balanced case as the distance between the two layers increases. ESF where  $\langle \psi \rangle \neq 0$ ,  $\langle n_{\vec{G}} \rangle = 0$  stands for excitonic superfluid, PSDW where  $\langle \psi \rangle = 0$ ,  $\langle n_{\vec{G}} \rangle \neq 0$  stands for pseudo-spin density wave phase, FL stands for Fermi Liquid. (a) Energy dispersion relation  $\omega(q)$  in these phases. (b)  $V_E(q)$  in these phases. The cross in the PSDW means the negative minimum value of  $V_E(q)$  is replaced by the PSDW. The order parameters are also shown. In fact, the instability happens before the minimum touches zero.

out the phase field in favor of the density operator. Neglecting the vortex excitations in  $\theta_-$  and integrating out the  $\theta_-$  in Eqn.3 leads to:

$$\mathcal{L}[\delta\rho_-] = \frac{1}{2}\delta\rho^-(-\vec{q}, -\omega_n)[\frac{\omega_n^2}{2\rho_E q^2} + V_E(\vec{q})]\delta\rho^-(\vec{q}, \omega_n) \quad (5)$$

where we can identify the dynamic pseudo-spin density-density correlation function  $S_-(\vec{q}, \omega_n) = \langle \delta\rho^-(-\vec{q}, -\omega_n)\delta\rho^-(\vec{q}, \omega_n) \rangle = \frac{2\rho_E q^2}{\omega_n^2 + v^2(q)q^2}$  where  $v^2(q) = 2\rho_EV_E(q)$  is the spin wave velocity.

From the pole of the dynamic density-density correlation function, we can identify the speed of sound wave which is exactly the same as the spin wave velocity. This should not be too surprising. As shown in liquid  $^4\text{He}$ , the speed of sound is exactly the same as the phonon velocity. Here, in the context of excitonic superfluid, we explicitly prove that the sound speed is indeed the same as the spin wave velocity. From the analytical continuation  $i\omega_n \rightarrow \omega + i\delta$ , we can identify the dynamic structure factor:  $S_-(\vec{q}, \omega) = S_-(q)\delta(\omega - v(q)q)$  where  $S_-(q) = \rho_E q\pi/v(q)$  is the equal time pseudo-spin correlation function shown in Fig.1. As  $q \rightarrow 0$ ,  $S_-(q) \rightarrow q$ . The Feymann relation in BLQH which relates the dispersion relation to the equal time structure factor is

$$\omega(q) = \frac{\rho_E \pi q^2}{S_-(q)} \quad (6)$$

which takes exactly the same form as the Feymann relation in superfluid  $^4\text{He}$ . Obviously, the  $V_E(q)$  in the Fig.1b leads to the magneto-roton dispersion  $\omega^2 = q^2V_E(q)$  in the Fig.1a.

Because the instability is happening at  $q = q_0$  instead of at  $q = 0$ , the vortex excitations in  $\theta_-$  remain *uncritical* through the transition. Integrating them out will not generate any singularities except the interactions among the pseudo-spin density  $\delta\rho_-$ . Expanding  $V_E(q)$  near the minimum  $q_0$  in the Fig. 1 leads to the quantum Ginsburg-Landau action to describe the ESF to the PSDW transition:

$$\mathcal{L}[\delta\rho_-] = \frac{1}{2}\delta\rho_-[A_\rho\omega_n^2 + r + c(q^2 - q_0^2)]\delta\rho_- + u(\delta\rho_-)^4 + v(\delta\rho_-)^6 + \dots \quad (7)$$

where the momentum and frequency conservation in the quartic and sixth order terms is assumed,  $A_\rho \sim \frac{1}{2\rho_E q_0^2}$  which is non-critical across the transition. While the corresponding quantity  $A_\theta \sim S_-(q)$  in the phase representation Eqn.3 is divergent, so Eqn.3 breaks down as  $q \rightarrow q_0^-$  and may not be used to describe the ESF to the PSDW transition.

In sharp contrast to the conventional classical normal liquid (NL) to normal solid (NS) transition [17], the possible cubic interaction term  $(\delta\rho_-)^3$  is forbidden by the  $Z_2$  exchange symmetry between the two layers  $\delta\rho_- \rightarrow -\delta\rho_-$ . Note that the  $\omega_n^2$  term in the first term stands for the quantum fluctuations of  $\delta\rho_-$  which is absent in the classical NL to NS transition. In the following section, we will show that the most favorable lattice is a square lattice.

**Lattice structure of the PSDW phase.** In Eqn.7,  $r$  which is the gap of  $V_E(q)$  at the minimum tunes the transition from the ESF to the PSDW. In the ESF,  $r > 0$  and  $\langle \delta\rho_- \rangle = 0$  is uniform. In the PSDW,  $r < 0$  and  $\langle \delta\rho_- \rangle = \sum_{\vec{G}} n(\vec{G})e^{i\vec{G}\cdot\vec{x}}$ ,  $n(0) = 0$  takes a lattice structure with reciprocal lattice vectors  $\vec{G}$ . It was shown that in the classical NL to NS transition, due to the cubic term, a triangular lattice is the favorite lattice. However, due to the absence of the cubic term in Eqn.7, in the following, we will show that the favorite lattice is the square lattice. At mean field level, we can ignore the  $\omega$  dependence in Eqn.7. Substituting  $\langle \delta\rho_- \rangle = \sum_{\vec{G}} n(\vec{G})e^{i\vec{G}\cdot\vec{x}}$  into Eqn.7 leads to:

$$f_n = \sum_{\vec{G}} \frac{1}{2} r_{\vec{G}} |n_{\vec{G}}|^2 + u \sum_{\vec{G}} n_{\vec{G}_1} n_{\vec{G}_2} n_{\vec{G}_3} n_{\vec{G}_4} \delta_{\vec{G}_1 + \vec{G}_2 + \vec{G}_3 + \vec{G}_4, 0} + v \sum_{\vec{G}} n_{\vec{G}_1} n_{\vec{G}_2} n_{\vec{G}_3} n_{\vec{G}_4} n_{\vec{G}_5} n_{\vec{G}_6} \delta_{\vec{G}_1 + \vec{G}_2 + \vec{G}_3 + \vec{G}_4 + \vec{G}_5 + \vec{G}_6, 0} \quad (8)$$

where  $u$  could be positive or negative,  $v > 0$  is always positive to keep the system stable.

From Eqn.8, we can compare the ground state energy of the two most commonly seen lattices: square lattice and triangular lattice. Square ( triangular ) lattice has  $m = 4$  (  $m = 6$  ) shortest reciprocal lattice vectors  $\vec{G} = 2\pi/a$  where  $a$  is the lattice constant in a given layer.

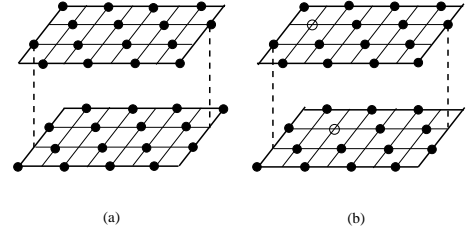


FIG. 2: (a) The charge distribution of the PSDW in a square lattice. The "up" pseudo-spins take sublattice A, while the "down" pseudo-spins take sublattice B. (b) In general, there are quantum and thermal fluctuation generated vacancies in both layers denoted by a  $\bigcirc$ .

Following [17], one can simplify Eqn.8 to:

$$f_\alpha = \frac{1}{2} r_{\vec{G}} |n_{\vec{G}}|^2 + u_\alpha |n_{\vec{G}}|^4 + v_\alpha |n_{\vec{G}}|^6 \quad (9)$$

where for the quadratic and quartic terms, all the contributions come from the paired reciprocal lattice vectors. After the scaling  $n_{\vec{G}} \rightarrow m^{-1/2} n_{\vec{G}}$ , the quadratic term become the same for both lattices, but the quartic terms are still different  $u_\alpha = 3(1 - 1/m)u$  [17], so  $u_\square = 2\frac{1}{4}u$ ,  $u_\Delta = 2\frac{1}{2}u$ . For the sixth order term, in square lattice, all the contributions are still from the paired reciprocal lattice vectors, however, in triangular lattice, there is an *additional* contribution from a triangle where  $\vec{G}_1 + \vec{G}_2 + \vec{G}_3 = 0$ . After very careful counting, we find that  $v_\alpha^p = 5(3 - 9/m + 8/m^2)v$  from the paired contribution and  $v^{tri} = 5/6v$  from the additional contribution from the triangle in a triangular lattice, so  $v_\square = 6\frac{1}{4}v$ ,  $v_\Delta = 9\frac{4}{9}v$ . The mean field phase diagram of Eqn.9 is well known: If  $u > 0$  (  $u < 0$  ), there is 2nd ( 1st ) order transition, there is a tri-critical point at  $u = 0$ . Minimizing  $f$  with respect to  $n_{\vec{G}}$  leads to  $n_\alpha^2 = (-2u_\alpha + \sqrt{4u_\alpha^2 - 6v_\alpha r})/6v_\alpha$  which holds for both  $u_\alpha > 0$  and  $u_\alpha < 0$ . Obviously,  $n_\square \neq n_\Delta$ . From Eqn.9, we can see that if  $u > 0$ , because  $u_\square < u_\Delta$ ,  $v_\square < v_\Delta$ , for any given  $n$ ,  $f_\square(n) < f_\Delta(n)$ , then  $f_\square(n_\square) < f_\square(n_\Delta) < f_\Delta(n_\Delta)$ . If  $u < 0$ , it is easy to show that when  $n^2 > n_c^2 = -\frac{9u}{108v}$ ,  $f_\square(n) < f_\Delta(n)$ . Because in the two lattice states,  $r < u_\alpha^2/2v_\alpha$ ,  $n_\square^2 > -\frac{u_\square}{2v_\square} = -\frac{9u}{50v} > n_c^2$ ,  $n_\Delta^2 > -\frac{u_\Delta}{2v_\Delta} = -\frac{9u}{68v} > n_c^2$ , so in this regime, we still have  $f_\square(n_\square) < f_\square(n_\Delta) < f_\Delta(n_\Delta)$ . So we conclude that in any case, the square lattice is the favorite lattice. It has three elastic constants instead of two. Neglecting zero-point quantum fluctuations,  $\langle \delta\rho_- \rangle = \sum_i \delta(\vec{x} - \vec{R}_i) - \sum_i \delta(\vec{x} - \vec{R}_i - \vec{l})$  where the  $\vec{l} = \frac{1}{2}(\vec{a}_1 + \vec{a}_2)$  is the shift of the square lattice in the bottom layer relative to that in the top layer ( Fig.2).

This PSDW state not only breaks the translational symmetry, but also the  $Z_2$  exchange symmetry. It is very rare to get a 2d square lattice, because it is not a close packed lattice. Due to the special  $Z_2$  symmetry of BLQH, we show it indeed can be realized in BLQH.

The system is compressible with gapless phonon excitations determined by the 3 elastic constants. It is very interesting to see if *in-plane* soft X-ray or light scattering experiments [2] can directly test the existence of the PSDW when  $d_{c1} < d < d_{c2}$ . Note that the light scattering intensity may be diminished by a Debye-Waller factor due to the large zero-point quantum fluctuations in the PSDW [16].

**Vacancies, disorders and Coulomb Drag in the PSDW state.** In principle, the  $\delta\rho_+$  mode in Eqn.2 should also be included. It stands for the translational (or sliding) motion of the PSDW lattice. However, any weak disorders will pin this PSDW state. Therefore, the  $\delta\rho_+$  mode can be neglected. Disorders will smear the 1st order transition from the ESF to the PSDW into a 2nd order transition. It was argued in [1] that disorders in real samples are so strong that they may even have destroyed the ESF state, so they may also transfer the long range lattice orders of the PSDW into short range ones. This fact makes the observation of the lattice structure by light scattering difficult. The two square lattices in the top and bottom layer are locked together, so it will show huge Coulomb drag. However, vacancies generated by the large zero-point quantum fluctuations may play important roles in the drag ( Fig.2b ). As the distance increases to the critical distance  $d_{c1}$  in Fig.1, the ESF turns into the PSDW whose lattice constant  $a = \sqrt{4\pi}l$  is completely fixed by the filling factor  $\nu_1 = 1/2$ , so it may not completely match the instability point  $1/q_0$ . Due to this slight mismatch, the resulting PSDW is very likely to be an *in-commensurate* solid where the total number of sites  $N_s$  may not be the same as the total number of sites  $N$  even at  $T = 0$ . As the distance increases further  $d_{c1} < d < d_{c2}$  in Fig.1, the PSDW lattice constant is still *locked* at  $a = \sqrt{4\pi}l$ . Assuming zero-point quantum fluctuations favor vacancies over interstitials [16], so there are vacancies  $n_0$  even at  $T = 0$  in each layer. At finite temperature, there are also thermally generated vacancies  $n_a(T) \sim e^{-\Delta_v/T}$  where  $\Delta_v$  is the vacancy excitation energy. So the total number of vacancies at any  $T$  is  $n_v(T) = n_0 + n_a(T)$ . Obviously, the vacancies in top layer are strongly correlated with those in the bottom layer. As shown in [18], the correlated character of hopping transports of the vacancies between the active and passive layers can lead to a very large drag. We can estimate the resistance in the active layer as  $R \sim 1/n_v(T)$ . As shown in [18], the drag resistance in the passive layer is  $R_D \sim \alpha_D R$  where  $\alpha_D$  should not be too small. This temperature dependence is indeed consistent with that found in the experiment [5]: starting from  $200mK$ ,  $R_D$  increases exponentially until to  $50mK$  and then saturates. This behavior is marked different than that at both small and large distance where the system is in the ESF and FL regimes respectively. These vacancies also lead to the finite Hall drag in the PSDW. We conclude that in the presence of disorders, all the properties of the

PSDW are consistent with the experimental observations in [5] on the intermediate phase. The effects of very small imbalance on the ESN phase was investigated in [11] and was also found to be consistent with the experimental data in [12].

**Discussions.** We compare the results achieved from the QGL theory with those achieved from the microscopic LLL+HF approximation in [6]. Especially, Cote, Brey and Macdonald in [6] numerically found that the lowest energy lattice structure of the PSDW is a square lattice. But it is not known if the HF+LLL calculations can describe the transition from the ESF to the PSDW. The QGL theory presented in this paper circumvents the difficulty associated with the unknown wavefunction at any finite  $d$  [19] and treat both the interlayer and the intralayer correlations on the same footing, so can be used to capture competing orders on microscopic length scales and naturally leads to the PSDW as the intermediate state which breaks translational symmetry. The theory puts the ESF state and the PSDW state on the same footing, characterize the symmetry breaking patterns in the two states by corresponding order parameters and describe the universality class of the quantum phase transition between the two states. We explicitly showed that the square lattice is the most favorite lattice. There are quantum fluctuation generated vacancies in the PSDW which lead to the unusual temperature dependence of the Coulomb drag observed in the experiment [5]. The QGL theory is complementary to and goes well beyond the previous microscopic calculations. Recent experiments [20] found the real spin also plays important roles in some BLQH samples with large tunneling gaps. Putting the spin into the theory remains an important open problem.

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